



6.4.3 Augmented matrices

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Introduction

- In this topic, we will
 - Define augmented matrices
 - Show how operations on systems of linear equations have a corresponding operation on the augmented matrix
 - Discuss the benefits of using augmented matrices versus systems of linear equations





Augmented matrices

- Let us start with either a system of linear equations, or equivalently, a linear combination of vectors equaling a target vector:

$$a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 + \cdots + a_{1,n}x_n = b_1$$

$$a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 + \cdots + a_{2,n}x_n = b_2$$

$$a_{3,1}x_1 + a_{3,2}x_2 + a_{3,3}x_3 + \cdots + a_{3,n}x_n = b_3$$

$$\vdots$$

$$a_{m,1}x_1 + a_{m,2}x_2 + a_{m,3}x_3 + \cdots + a_{m,n}x_n = b_m$$

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{pmatrix} \quad \mathbf{a}_j = \begin{pmatrix} a_{1,j} \\ a_{2,j} \\ a_{3,j} \\ \vdots \\ a_{m,j} \end{pmatrix}$$





Matrix representation

- Given the system of linear equations, we defined the matrix A
 - Let us *augment* the matrix by appending the vector $\mathbf{b}$:

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & a_{m,3} & \cdots & a_{m,n} \end{pmatrix}$$

The dotted line separates the coefficients and the right-hand values

$$\left(A \mid \mathbf{b} \right) = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} & b_1 \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} & b_2 \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n} & b_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m,1} & a_{m,2} & a_{m,3} & \cdots & a_{m,n} & b_m \end{pmatrix}$$





Matrix representation

- While matrices will have a significance beyond the coefficients, the **only** purpose of the augmented matrix is to help us systematically find a solution to our problem
 - It is a *data structure*, not a mathematical object

$$(A \mid \mathbf{b}) = (\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3 \ \cdots \ \mathbf{a}_n \mid \mathbf{b}) = \left(\begin{array}{ccccc|c} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} & b_1 \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} & b_2 \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n} & b_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m,1} & a_{m,2} & a_{m,3} & \cdots & a_{m,n} & b_m \end{array} \right)$$





Matrix representation

- Suppose we have a system of linear equation we're trying to solve:

$$\begin{array}{rcl} x_1 + 3x_2 & = & -2 \\ 4x_1 + 7x_2 & = & -5 \end{array} \qquad \left(\begin{array}{cc|c} 1 & 3 & -2 \\ 4 & 7 & -5 \end{array} \right)$$

- Adding -4 times Eqn 1 onto Eqn 2 yields:

$$\begin{array}{rcl} x_1 + 3x_2 & = & -2 \\ -5x_2 & = & 3 \end{array} \qquad \left(\begin{array}{cc|c} 1 & 3 & -2 \\ 0 & -5 & 3 \end{array} \right)$$

- Instead, add -4 times Row 1 onto Row 2





Matrix representation

- As a second example

$$\begin{array}{l} -x_1 - 2x_2 + 3x_3 + 8x_4 = -3 \\ 3x_1 - 4x_2 + 5x_3 - 6x_4 = 7 \end{array} \quad \left(\begin{array}{cccc|c} -1 & -2 & 3 & 8 & -3 \\ 3 & -4 & 5 & -6 & 7 \end{array} \right)$$

- Adding 3 times Eqn 1 onto Eqn 2 yields:

$$\begin{array}{l} -x_1 - 2x_2 + 3x_3 + 8x_4 = -3 \\ -10x_2 + 14x_3 + 18x_4 = -2 \end{array} \quad \left(\begin{array}{cccc|c} -1 & -2 & 3 & 8 & -3 \\ 0 & -10 & 14 & 18 & -2 \end{array} \right)$$

- We could have added 3 times Row 1 onto Row 2





Use of augmented matrices

- We will subsequently reinterpret all of the steps taken in finding a solution to a system of linear equations to manipulations of an augmented matrix to find a solution to the corresponding system of linear equations
 - Row operations
 - Backward substitution
 - Free variables
- This will allow us to program these algorithms into a computer





Use of augmented matrices

- As a side note, if we find a solution the system of linear equations corresponding to this augmented matrix

$$\left(\begin{array}{cc|c} 1 & 3 & -2 \\ 4 & 7 & -5 \end{array} \right)$$

we get $\mathbf{x} = \begin{pmatrix} -0.2 \\ -0.6 \end{pmatrix}$

- What this means is that

$$-0.2 \begin{pmatrix} 1 \\ 4 \end{pmatrix} - 0.6 \begin{pmatrix} 3 \\ 7 \end{pmatrix} = \begin{pmatrix} -2 \\ -5 \end{pmatrix}$$





Summary

- Following this topic, you now
 - Understand how to get an augmented matrix
 - Know that we can perform equivalent operations to what we did with equations on the augmented matrix
 - Are aware that it will be easier to describe and program such an algorithm without the variables
 - Understand that a solution to the system of linear equations gives the same solution to the question of finding a linear combination vectors equaling a target vector





References

- [1] https://en.wikipedia.org/wiki/Augmented_matrix





Acknowledgments

None so far.





Colophon

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The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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